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# THICKNESS-SHEAR AND FLEXURAL VIBRATIONS OF A CIRCULAR DISK

by

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## ABSTRACT

Equations governing the coupled thickness-shear and flexural vibrations of isotropic, elastic plates are solved for the case of a circular disk with free edges. The results of computations are given for the resonant frequencies analogous to those which would be excited in an AT-cut quartz disk with full electrode coating. A marked difference is observed between the frequency spectrum of the disk and the spectrum of cylindrical vibrations of a rectangular plate. The difference is due to the presence of thickness-twist modes of motion, in the disk, which, along with the thickness-shear modes, are coupled to the flexural modes.

## Introduction

This is the seventh in a series of papers on vibrations of plates. The first paper[1] contained a derivation of isotropic plate equations which accommodate the phenomenon of coupling between flexural and thickness-shear modes of motion. These equations were later extended to take into account crystal plates[2], piezoelectric crystal plates, plates of varying thickness, plates with incomplete electrodes and plates with specially shaped electrodes. (The first[1] and second[2] papers are referred to as I and II in the sequel.) The solutions of the various equations, given as illustrative examples in these papers, were all confined to cylindrical motions, that is, to one-dimensional problems.

We return, now, to the simplest (isotropic) form of the equations to solve a two-dimensional problem. We have in mind the antisymmetric modes of motion that are excited in an AT-cut quartz disk. However, the additional mathematical complexities introduced by the circular shape of the plate make it advisable to consider, first, the corresponding motions of an isotropic disk. It seems likely that the modes of motion and the frequency spectrum of the isotropic disk will be found to differ insignificantly from those of the quartz disk. This is because the two controlling parameters are the shape of the plate and the ratio of the plate-flexure modulus to the thickness-shear modulus. The former

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1 R. D. Mindlin, J. Appl. Mech. 18, 31-38 (1951).
2 R. D. Mindlin, J. Appl. Phys. 22, 316-323 (1951).

is the same for the two cases, and the latter has been made identical, in the computations, simply by choosing Poisson's ratio, for the isotropic material, to give the modulus ratio appropriate to the AT-cut of quartz. The lateral elastic constants, it is true, are not quite the same, but the discrepancy is small, and its influence on the motion is probably slight.

We give, here, the exact solutions of the equations for the isotropic, free disk and the results of detailed computations of the frequencies in the neighborhood of the thickness-shear frequency of an infinite plate. The frequency spectrum of the disk is found to have a different character from that obtained in II for the cylindrical vibrations of a rectangular plate. At first glance, it appears that the thickness-shear fundamental and its overtones are multi-valued, i.e., corresponding to the fundamental and to each overtone of thickness-shear in the rectangular plate, there appears to be more than one resonant frequency in the disk. Upon closer examination, it develops that the additional resonances are due to the presence of a thickness-twist mode and its overtones, in which the displacement is a twist about an axis normal to the plate. The complete frequency spectrum is the result of coupling between thickness-shear and flexure and between thickness-twist and flexure.

## Plate Equations

In I and II, the plate equations were deduced from the three-dimensional equations of elasticity, referred to a rectangular coordinate system. The resulting equations may be transformed to a polar coordinate system, appropriate to the circular disk, but it is simpler to rederive the plate equations in the new system of coordinates. Thus, the assumed form of the displacements, in place of I(10) or II(5), is

$$u_r \approx y\,\psi_r(r,\theta,t), \qquad u_\theta \approx y\,\psi_\theta(r,\theta,t), \qquad u_y \approx \eta(r,\theta,t) \tag{1}$$

where $r$ and $\theta$ are polar coordinates defined by $x = r\cos\theta$, $z = r\sin\theta$, i.e., the $y$ axis is normal to the plane of the plate.

The derivation of the plate equations proceeds as before. Thus, corresponding to I(14,15) or II(10), we have plate-stress equations of motion

$$\begin{aligned}
\frac{\partial M_r}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{M_r - M_\theta}{r} - Q_r &= \frac{\rho h^3}{12}\frac{\partial^2 \psi_r}{\partial t^2} \\
\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_\theta}{\partial \theta} + \frac{2M_{r\theta}}{r} - Q_\theta &= \frac{\rho h^3}{12}\frac{\partial^2 \psi_\theta}{\partial t^2} \\
\frac{\partial Q_r}{\partial r} + \frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{Q_r}{r} &= \rho h \frac{\partial^2 \eta}{\partial t^2}
\end{aligned} \tag{2}$$

where $\rho$ is the density, $h$ is the thickness of the plate and the plate-stress components are given in terms of the plate-displacements by

$$\begin{aligned}
M_r &= D\left[\frac{\partial \psi_r}{\partial r} + \frac{\nu}{r}\left(\psi_r + \frac{\partial \psi_\theta}{\partial \theta}\right)\right] \\
M_\theta &= D\left[\frac{1}{r}\left(\psi_r + \frac{\partial \psi_\theta}{\partial \theta}\right) + \nu\frac{\partial \psi_r}{\partial r}\right] \\
M_{r\theta} &= \frac{D}{2}(1-\nu)\left[\frac{1}{r}\left(\frac{\partial \psi_r}{\partial \theta} - \psi_\theta\right) + \frac{\partial \psi_\theta}{\partial r}\right] \\
Q_r &= \kappa^2 \mu h \left(\psi_r + \frac{\partial \eta}{\partial r}\right) \\
Q_\theta &= \kappa^2 \mu h \left(\psi_\theta + \frac{1}{r}\frac{\partial \eta}{\partial \theta}\right)
\end{aligned} \tag{3}$$

where $D = Eh^3/12(1-\nu^2)$ , $E$, $\mu$, $\nu$ are Young's modulus, the shear modulus and Poisson's ratio, respectively, and $\kappa^2 = \pi^2/12$.

If (3) were substituted in (2) the resulting plate-displacement equations of motion would be those corresponding to I(16) or II(11). However, in the present case it is more convenient to work with the alternative formulation of the equations given by I(62,63). In polar coordinates, again omitting a factor $e^{ipt}$, these become

$$\psi_r = (\sigma_1 - 1)\frac{\partial \eta_1}{\partial r} + (\sigma_2 - 1)\frac{\partial \eta_2}{\partial r} + \frac{1}{r}\frac{\partial H}{\partial \theta}$$

$$\psi_\theta = (\sigma_1 - 1)\frac{\partial \eta_1}{r\partial \theta} + (\sigma_2 - 1)\frac{\partial \eta_2}{r\partial \theta} - \frac{\partial H}{\partial r} \qquad (4)$$

$$\eta = \eta_1 + \eta_2$$

$$(\nabla^2 + \delta_1^2)\eta_1 = 0$$

$$(\nabla^2 + \delta_2^2)\eta_2 = 0 \qquad (5)$$

$$(\nabla^2 + \omega^2) H = 0$$

where

$$\delta_1^2, \delta_2^2 = \frac{1}{2}\delta_0^4\left[R + S \pm \sqrt{(R-S)^2 + 4\delta_0^{-4}}\right] \qquad (6)$$

$$\sigma_1, \sigma_2 = (\delta_2^2, \delta_1^2)(R\delta_0^4 - S^{-1})^{-1} \qquad (7)$$

$$\omega^2 = 2(R\delta_0^4 - S^{-1})/(1-\nu) \qquad (8)$$

$$R = h^2/12, \quad S = D/\kappa^2\mu h, \quad \delta_0^4 = \rho p^2 h/D \qquad (9)$$

## Solution of Equations of Motion

In view of the modes of motion of interest in the present case, the appropriate solutions of (5) are

$$\begin{aligned} \eta_1 &= A_1 J_1(\delta_1 r) \cos\theta \\ \eta_2 &= A_2 J_1(\delta_2 r) \cos\theta \\ H &= A_3 J_1(\omega r) \sin\theta \end{aligned} \tag{10}$$

where $J_1(x)$ is Bessel's function of the first kind, of order unity, and $A_1$, $A_2$, $A_3$ are arbitrary constants. These are the modes of circumferential order unity. There are also modes of higher and lower circumferential order in which, for example, $\eta_1 = A_1 J_n(\delta_1 r) \cos n\theta$, $n = 0, 1, 2, \cdots$, but we are not concerned with them here.

For the vibrations of a plate with free edges, the boundary conditions are

$$M_r = M_{r\theta} = Q_r = 0 \qquad \text{at} \qquad r = a \tag{11}$$

where $a$ is the radius of the disk. Using (3), (4) and (10), Equations (11) become

$$\left[\left(2-\frac{\delta_1^2 a^2}{1-\nu}\right) J_1(\delta_1 a) - \delta_1 a J_0(\delta_1 a)\right](\sigma_1 - 1) A_1$$

$$+ \left[\left(2-\frac{\delta_2^2 a^2}{1-\nu}\right) J_1(\delta_2 a) - \delta_2 a J_0(\delta_2 a)\right](\sigma_2 - 1) A_2$$

$$+ \left[\omega a J_0(\omega a) - 2 J_1(\omega a)\right] A_3 = 0$$

$$\left[\delta_1 a J_0(\delta_1 a) - 2 J_1(\delta_1 a)\right](\sigma_1 - 1) A_1$$

$$+ \left[\delta_2 a J_0(\delta_2 a) - 2 J_1(\delta_2 a)\right](\sigma_2 - 1) A_2 \qquad (12)$$

$$+ \left[\left(2-\frac{\omega^2 a^2}{2}\right) J_1(\omega a) - \omega a J_0(\omega a)\right] A_3 = 0$$

$$\left[\delta_1 a J_0(\delta_1 a) - J_1(\delta_1 a)\right]\sigma_1 A_1$$

$$+ \left[\delta_2 a J_0(\delta_2 a) - J_1(\delta_2 a)\right]\sigma_2 A_2$$

$$+ J_1(\omega a) A_3 = 0$$

Frequency Equation

Equations (12) comprise a system of three homogeneous, algebraic equations on the unknowns $A_1$, $A_2$, $A_3$. The vanishing of the determinant of their coefficients yields an equations for the determination of the frequency $p$ through $\delta_1$, $\delta_2$ and $\omega$. This secular equation, expressed in a form convenient for desk-calculator computation, reads:

$$
(1-\beta^2)\gamma^2\Gamma_1\Gamma_2 + \frac{1-g\beta^2}{1+g}\left[\frac{2(1+\beta^2)}{(1-\nu)(1+g)}\right]^{1/2}\gamma^2\Gamma_1\Gamma_3
$$

$$
-\frac{1}{\beta}\frac{1-g\beta^2}{1+g}\left(3+\frac{1+\beta^2}{1+g}+\frac{g-\beta^2}{1-g\beta^2}-\frac{\beta^2}{1-\nu}\gamma^2\right)\gamma\Gamma_1
$$

$$
+\frac{1}{\beta}\frac{g-\beta^2}{1+g}\left[\frac{2(1+\beta^2)}{(1-\nu)(1+g)}\right]^{1/2}\gamma^2\Gamma_2\Gamma_3
$$

$$
-\frac{g-\beta^2}{1+g}\left(3+\frac{1}{\beta^2}\frac{1+\beta^2}{1+g}+\frac{1-g\beta^2}{g-\beta^2}-\frac{\gamma^2}{1-\nu}\right)\gamma\Gamma_2 \tag{13a}
$$

$$
-\frac{1-g\beta^2}{1+g}\left(1+\frac{1}{\beta^2}\frac{g-\beta^2}{1-g\beta^2}\right)\left[\frac{2(1+\beta^2)}{(1-\nu)(1+g)}\right]^{1/2}\gamma\Gamma_3
$$

$$
+\frac{1-\beta^2}{\beta}\left[4-\frac{g(1+\beta^2)}{(1-\nu)(1+g)}\gamma^2\right]=0
$$

where

$$
\beta=\delta_2/\delta_1, \qquad \gamma=\delta_1 a, \qquad g=R/S
$$

$$
\Gamma_i=J_0(c_i\gamma)/J_1(c_i\gamma), \qquad i=1,2,3
$$

$$
c_1=1, \qquad c_2=\beta, \qquad c_3=\beta\sqrt{\frac{2(1+g)}{(1-\nu)(1+\beta^2)}}
$$

Now, $\delta_1^2>0$ for all values of $p>0$ while $\delta_2^2 \gtreqless 0$ according as $p \gtreqless \bar{p}$, where $\bar{p}=\pi(\mu/\rho)^{1/2}/h$ is the frequency of the first thickness-shear mode of an infinite plate. Hence, $\beta$ will be real or imaginary according as $p \gtrless \bar{p}$. Equation (13a) is a useful form for $p>\bar{p}$. In the range $p<\bar{p}$, let $\beta=i\beta_1$, so that the frequency equation reads

$$
\begin{aligned}
&(1+\beta_1^2)\gamma^2\Gamma_1 G_2 + \frac{1+g\beta_1^2}{1+g}\left[\frac{2(1-\beta_1^2)}{(1-\nu)(1+g)}\right]^{1/2}\gamma^2\Gamma_1 G_3 \\
&-\frac{1}{\beta_1}\frac{1+g\beta_1^2}{1+g}\left(3+\frac{1-\beta_1^2}{1+g}+\frac{g+\beta_1^2}{1+g\beta_1^2}+\frac{\beta_1^2}{1-\nu}\gamma^2\right)\gamma\Gamma_1 \\
&-\frac{1}{\beta_1}\frac{\beta_1^2+g}{1+g}\left[\frac{2(1-\beta_1^2)}{(1-\nu)(1+g)}\right]^{1/2}\gamma^2 G_2 G_3 \\
&-\frac{\beta_1^2+g}{1+g}\left(3-\frac{1}{\beta_1^2}\frac{1-\beta_1^2}{1+g}+\frac{1+g\beta_1^2}{g+\beta_1^2}-\frac{\gamma^2}{1-\nu}\right)\gamma G_2 \\
&+\frac{1+g\beta_1^2}{1+g}\left(-1+\frac{1}{\beta_1^2}\frac{\beta_1^2+g}{1+g\beta_1^2}\right)\left[\frac{2(1-\beta_1^2)}{(1-\nu)(1+g)}\right]^{1/2}\gamma G_3 \\
&+\frac{1+\beta_1^2}{\beta_1}\left[4-\frac{g(1-\beta_1^2)}{(1-\nu)(1+g)}\gamma^2\right] = 0
\end{aligned}
\tag{13b}
$$

where

$$G_2 = I_0(\beta_1\gamma)/I_1(\beta_1\gamma)$$

$$G_3 = I_0(c'\gamma)/I_1(c'\gamma)$$

$$c' = \beta_1\sqrt{\frac{2(1+g)}{(1-\nu)(1-\beta_1^2)}}$$

and $I_0(x)$, $I_1(x)$ are modified Bessel's functions of the first kind.

From the relation $\beta = \delta_2/\delta_1$ and Equation (6), explicit expressions for the frequency may be obtained. Thus,

$$p/\bar{p} = \left[1 - \beta^2(1+g)^2/g(1+\beta^2)^2\right]^{-1/2}, \qquad p > \bar{p}$$

$$p/\bar{p} = \left[1 + \beta_1^2(1+g)^2/g(1-\beta_1^2)^2\right]^{-1/2}, \qquad p < \bar{p} \tag{14}$$

Further, from the relation $\gamma = \delta_1 a$,

$$d/h = \gamma(\bar{p}/p)\left[(1+\beta^2)/3(1+g)\right]^{1/2}, \qquad p > \bar{p}$$

$$d/h = \gamma(\bar{p}/p)\left[(1-\beta_1^2)/3(1+g)\right]^{1/2}, \qquad p < \bar{p} \tag{15}$$

where $d$ is the plate diameter.

## Computation of Frequencies

Equations (13), (14) and (15) constitute the solution of the problem. For a given material the values of $\nu$ and $g$ are fixed. Choosing a value of $\beta$ or $\beta_1$ fixes $p/\bar{p}$ by (14) and determines an infinite set of roots, $\gamma$, of (13). For the particular $\beta$ or $\beta_1$ each of these roots determines a ratio $d/h$, thus giving an infinite set of $d/h$ corresponding to every value of $p/\bar{p}$.

For the isotropic plate, the ratio of the thickness-shear modulus to the plate flexure modulus is

$$g = \pi^2(1-\nu)/24$$

while, for the AT-cut of quartz,

$$g = \frac{\pi^2 c_{66}}{12(c_{11} - c_{12}^2/c_{22})}$$

where the elastic constants are referred to the coordinates of the (rotated) AT section. For this cut of quartz, $g = 0.283$. Hence, an isotropic plate with $\nu = 0.312$ gives the same value of $g$ as an AT quartz plate. The computations were made for this value of $\nu$.

To facilitate the task of finding the roots $\gamma$ of Equations (13), for a series of values of $p/\bar{p}$, tables were computed of the functions:

| | |
|---|---|
| $J_0(x)/J_1(x)$ | 0(.01)100 |
| $I_0(x)/I_1(x)$ | 0(.01)10, 10(.02)16, 16(.1)20, 20(1)37 |

to six decimals.[3] The subsequent numerical work, carried out on a desk calculator, yielded the resonant frequencies of vibration depicted in Fig. 1.

The flexural modes, in the lower part of Fig. 1, are those of even order $r = 2, 4, 6 \ldots$, and are similar to those in the lower part of Fig. 1 of II. However, the upper parts ($p/\bar{p} > 1$) of the two figures are different. For example, in the case of the rectangular plate the thickness-shear fundamental couples with every even flexural mode in turn, but, in the disk, the thickness-shear fundamental appears to couple only with alternate, even, flexural modes. As a result, the thickness-shear fundamental appears to be double-valued.

The apparent anomaly is resolved when it is recognized that there is present a set of thickness-twist modes in addition to thickness-shear and flexural modes. Mathematically, the additional modes derive from the solenoidal, retarded potential $H$. Their uncoupled frequencies may be

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3 The computation of the tables of $J_0(x)/J_1(x)$ was performed by S. Poley on equipment made available by the Watson Scientific Computing Laboratory of the International Business Machine Corporation. Both tables are stored in the files of the Department of Civil Engineering, Columbia University.

obtained, approximately, by setting $\eta = 0$, in the equations, so that $H$ is the only potential left. Then we have

$$(\nabla^2 + \omega^2) H = 0$$

$$\psi_r = \frac{1}{r}\frac{\partial H}{\partial \theta} , \quad \psi_\theta = -\frac{\partial H}{\partial r}$$

$$M_{r\theta} = -\frac{D(1-\nu)}{2}\left(2\frac{\partial^2}{\partial r^2} + \omega^2\right) H$$

$$= -D(1-\nu)\,\omega^2 A_3 f(\omega r)\sin\theta$$

where

$$f(\omega r) = \left(\frac{2}{\omega^2 r^2} - \frac{1}{2}\right) J_1(\omega r) - \frac{1}{\omega r} J_0(\omega r)$$

Applying only the boundary condition $M_{r\theta} = 0$ on $r = a$, we obtain the secular equation $f(\omega a) = 0$, or

$$2c_3\gamma\Gamma_3 = 4 - c_3^2\gamma^2 \qquad (16)$$

The roots of (16), in conjunction with (14) and (15), give the approximate frequency spectrum ( $p/\bar{p}$ vs. $d/h$ ) of the thickness-twist modes. The curves are asymptotic to $p/\bar{p} = 1$ and are very nearly tangent to the almost horizontal portions of the curves in Fig. 1. That is, the thickness-twist modes have frequencies very close to those of the corresponding thickness-shear modes. The effect, of two such neighboring families, on the coupling with the flexural modes is illustrated in Fig. 2. In Fig. 2a is shown the usual spectrum resulting from the coupling of a family of odd thickness-shear modes ( $n$ = 1,3 ... ) and even flexural modes ( $r$ = 20,22 ... ). In Fig. 2b a family of thickness-twist modes ( $q$ = 1,3 ... ), closely paralleling the thickness-shear modes, is added. If the result were to be interpreted as coupling between flexure and thickness-shear alone, it would appear that the thickness-shear modes are double-valued and couple only with alternate flexure modes. This would be especially so if the coupling were strong enough to smooth out the kinks in the almost horizontal portions of the curves, as is the case in Fig. 1. However, when the presence of the thickness-twist modes is taken into account, there are no anomalies.

It should be observed that the case described here is a degenerate one of coupling of three families of modes, because, although there is coupling between thickness-shear and flexure and between thickness-twist and flexure, there is no coupling between thickness-shear and thickness-twist. An example in which there is full coupling of the three families will be described in a subsequent paper.

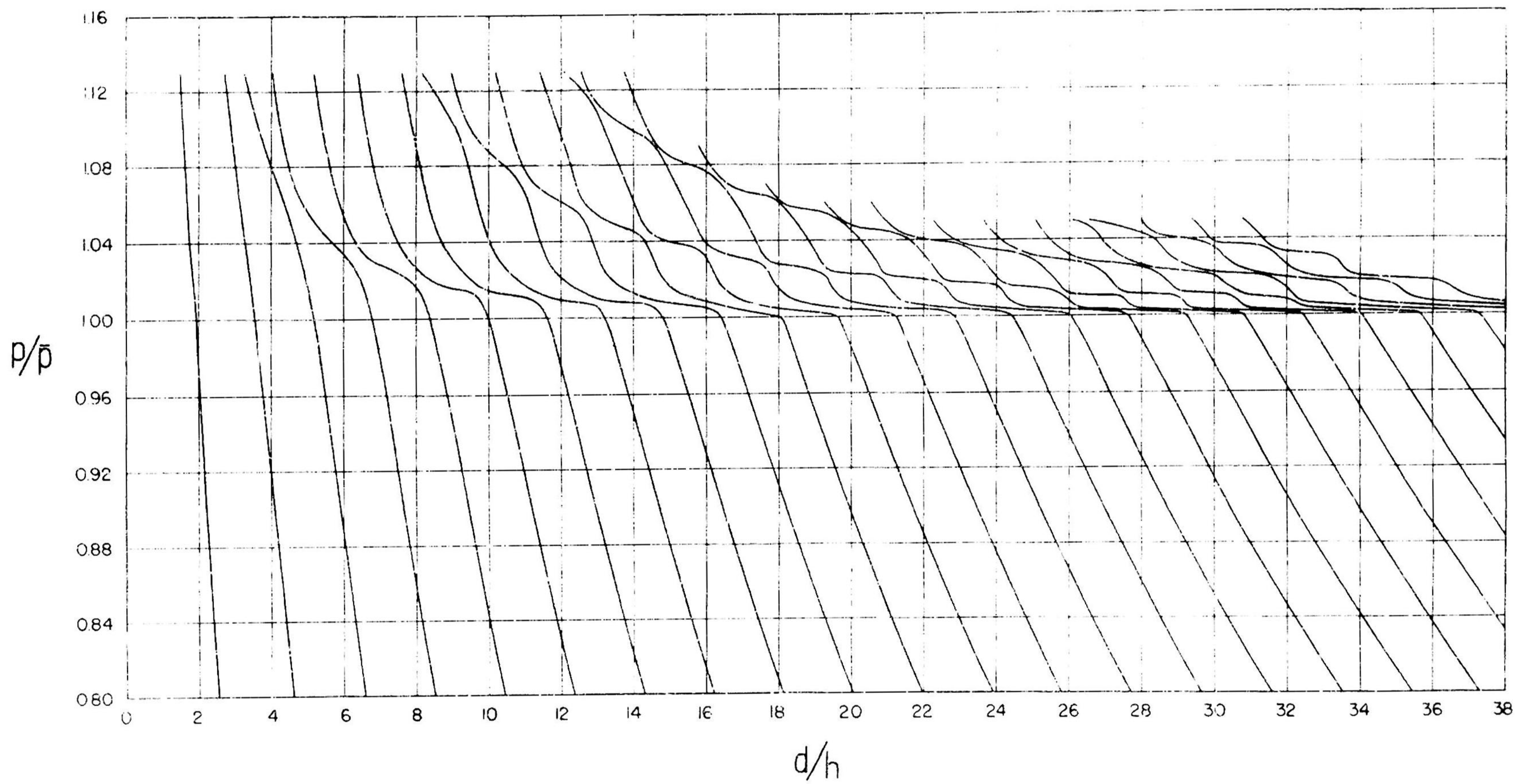


**Fig. 1:** **Resonant frequencies of modes of vibration of circumferential order unity of a circular disk** ( $\nu = 0.312$ ) as computed from Eqs. (13), (14), and (15). The ordinate is the ratio of the resonant frequency ( $p$ ) to the frequency ( $\bar{p}$ ) of the first thickness-shear mode of an infinite plate. The abscissa is the ratio of the diameter ( $d$ ) to the thickness ( $h$ ) of the disk.

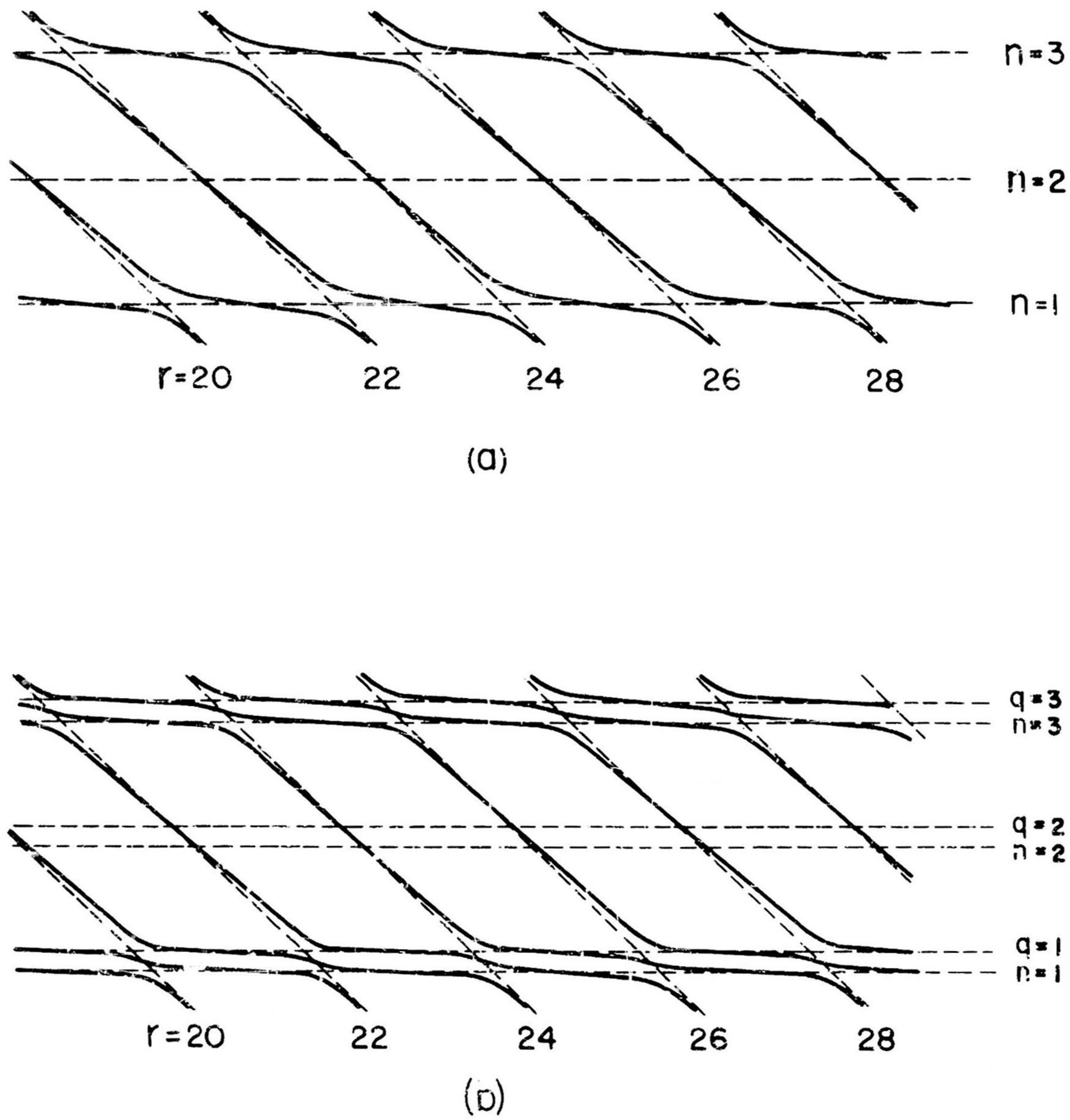


Fig. 2: (a) Frequency spectrum for coupling between thickness-shear and flexural modes.

(b) Frequency spectrum for coupling between thickness-shear and flexural modes and between thickness-twist and flexural modes, with no coupling between shear and twist modes.